A UNIQUE DECOMPOSITION RESULT FOR HT FACTORS WITH TORSION FREE CORE

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Abstract. We prove that II_1 factors M have a unique (up to unitary conjugacy) cross-product type decomposition around "core subfactors" $N \subset M$ satisfying the property HT of ([P1]) and a certain "torsion freeness" condition. In particular, this shows that isomorphism of factors of the form $L_\alpha(\mathbb{Z}^2) \rtimes \Gamma$, for torsion free, non-amenable subgroups $\Gamma \subset SL(2, \mathbb{Z})$ and $\alpha = e^{2\pi i t}$, $t \notin \mathbb{Q}$, implies isomorphism of the corresponding groups Γ.

Let M be a type II_1 factor and $B \subset M$ a von Neumann subalgebra. Recall from (Section 2 in [P1]) that M has the *property* H *relative to* B if the identity map on M can be approximated pointwise by regular (i.e., subunital, subtracial) completely positive maps on M that "vanish at infinity" relative to B. Also, $B \subset M$ is a *rigid inclusion* (or (M, B) has the *relative property* (T)) if regular completely positive maps on M that are close (pointwise) to the identity on M follow uniformly close to the identity on B (cf. 4.2 in [P1]). If $B \subset M$ satisfies both these conditions, then it is called a *HT inclusion* and B is called a *HT subalgebra* of M. (N.B. In fact, the notation used in [P1] to designate this property is "HT_s", while "HT" is used for a slightly weaker condition. We opted for the notation "HT" in this paper for the benefit of simplicity.) We refer to ([P1]) for the detailed definitions, as well as for notations and terminology used hereafter.

In ([P1]), one primarily studies HT Cartan subalgebras of M, i.e., HT inclusions $B \subset M$ with B maximal abelian in M and with the normalizer $\mathcal{N}_M(B) = \{u \in \mathcal{U}(M) \mid uBu^* = B\}$ generating M. Thus, the main technical result in ([P1]) shows the uniqueness, up to unitary conjugacy, of the HT Cartan subalgebras in separable II_1 factors. Such unique decomposition results are crucial for the calculation of invariants of II_1 factors (e.g., the fundamental and automorphism groups) and ultimately for their classification. Thus, one application of the uniqueness result for HT Cartan

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subalgebras in ([P1]) shows that two cocycle group von Neumann factors of the form $L_{\mu_{\alpha_j}}(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}))$, with μ_{α_j} the $SL(2,\mathbb{Z})$-invariant cocycle on $\mathbb{Z}^2$ given by "rational rotation" $\alpha_j = e^{2\pi i t_j}, t_j \in \mathbb{Q}$, $j = 1, 2$, are isomorphic if and only if the rational numbers t_1, t_2 have the same denominator.

In this paper we prove a new unique decomposition result, this time for HT inclusions $B \subset M$ with factorial "core" subalgebras, $B = N$, which satisfy the following "strong irreducibility" condition:

Definition. An inclusion $N \subset M$ is *torsion free* if N is a factor and $Q' \cap M = \mathbb{C}$ for any subfactor $Q \subset N$ with $[N : Q] < \infty$ and $Q' \cap N = \mathbb{C}$. By a Proposition below, an inclusion of the form $N \subset M = N \rtimes_\sigma \Gamma$ is torsion free iff N is a factor, the action σ is properly outer and the group Γ is torsion free, a fact that justifies the terminology.

If $N \subset M$ is torsion free and HT, then we also say that N is a *torsion free* HT *core* for M. We denote by $\mathcal{HT}_{tf}$ the class of factors that admit torsion free HT cores. The following are examples of such (inclusions of) factors:

Example 1. Let $M_\alpha(\Gamma)$ denote the factors defined as in (3.3.2 and 6.9.1 of [P1]), i.e., $M_\alpha(\Gamma) = R_\alpha \rtimes_{\sigma_\alpha} \Gamma$, where $\alpha = e^{2\pi i t}$ for some $t \in [0, 1/2] \setminus \mathbb{Q}$, R_α is the hyperfinite II_1 factor represented as the "irrational rotation algebra" $L_\alpha(\mathbb{Z}^2)$, $\Gamma \subset SL(2,\mathbb{Z})$ is a non-amenable (equivalently non-solvable) group and σ_α is the action of Γ on $L_\alpha(\mathbb{Z}^2)$ induced by the action of $SL(2,\mathbb{Z})$ on $\mathbb{Z}^2$. Note that by (page 62 of [Bu]) a group $\Gamma \subset SL(2,\mathbb{Z})$ is non-amenable if and only if the pair $(\mathbb{Z}^2 \rtimes \Gamma, \mathbb{Z}^2)$ has the relative property (T) ([M]). Thus, by (5.1 in [P1]) the inclusions $R_\alpha \subset M_\alpha(\Gamma)$ are rigid and since all $\Gamma \subset SL(2,\mathbb{Z})$ have Haagerup's compact approximation property, by (3.1 in [P1]), $M_\alpha(\Gamma)$ has property H relative to R_α. Moreover, by (3.3.2 (*ii*) in [Betti]) σ_α are properly outer actions (because the action of Γ on $\mathbb{Z}^2$ is outer) and by (3.3.2 (*i*) in [Betti]) they are ergodic as well. Indeed, this is because the stabilizer of any non-zero element in $\mathbb{Z}^2$ is a cyclic subgroup of $SL(2,\mathbb{Z})$, so if Γ leaves a finite subset $\neq \{(0,0)\}$ of $\mathbb{Z}^2$ invariant, then it is almost cyclic.

Altogether, this shows that $R_\alpha \subset M_\alpha(\Gamma)$ are irreducible HT inclusions of II_1 factors. The algebras $M_\alpha(\Gamma)$, will be called *irrational rotation* HT *factors*. Note that the inclusion $R_\alpha \subset M_\alpha(\Gamma)$ is torsion free whenever the non-amenable group $\Gamma \subset SL(2,\mathbb{Z})$ is torsion free. In particular one can take $\Gamma = \mathbb{F}_n$ for any $2 \leq n \leq \infty$.

More generally, if Γ is a torsion free Haagerup group acting outerly on a group H such that the pair $(H \rtimes \Gamma, H)$ has the relative property (T), and if μ is a Γ-invariant scalar 2-cocycle on H such that $L_\mu(H)$ is a factor, then $L_\mu(H) \rtimes \Gamma \in \mathcal{HT}_{tf}$, with $L_\mu(H)$ a torsion free HT core for $L_\mu(H) \rtimes \Gamma$. The irrational rotation factors $M_\alpha(\Gamma)$ correspond to the case the cocycle $\mu = \mu_\alpha$ is given by $\alpha = e^{2\pi i t}$, $t \notin \mathbb{Q}$.

Example 2. Let N be a type II_1 factor with the property (T) of Connes-Jones ([C1,2], [CJ]). If β is an aperiodic automorphism of N and one still denotes by β the properly

outer action of $\mathbb{Z}$ on N implemented by $\{\beta^n\}_{n\in\mathbb{Z}}$, then $N \subset N \rtimes_\beta \mathbb{Z}$ is a torsion free HT inclusion. More generally, any properly outer cocycle action $\sigma : \Gamma \to \mathrm{Aut}(N)$ of a torsion free group Γ satisfying Haagerup's compact approximation property gives rise to a torsion free HT inclusion $N \subset N \rtimes_\sigma \Gamma$. Indeed, by (4.7 and 5.9 in [P1]) $N \subset M$ is rigid, while M has the property H relative to N by (3.1 in [P1]). Note that these examples include some of the factors considered in the last part of Example 1: Thus, if H is taken to be an ICC group with the property (T) of Kazhdan and Γ a torsion free Haagerup group of outer automorphisms of H (e.g., $\Gamma = \mathbb{Z}$), then $N = L(H) \subset L(H \rtimes \Gamma) = M$ is a torsion free HT inclusion. More generally, one can take H arbitrary with the property (T), but with a scalar Γ-invariant cocycle μ on it such that $N = L_\mu(H)$ is a factor.

Example 3. If M is a II_1 factor with torsion free HT core $N \subset M$ and $t > 0$ then the amplification M^t of M has N^t as HT core (by 2.4 and 4.7 in [P1]), which trivially follows torsion free. Also, if $N_i \subset M_i = N_i \rtimes \Gamma_i, i = 1, 2$, are cross-product torsion free HT inclusions then $N_1 \overline{\otimes} N_2 \subset M_1 \overline{\otimes} M_2$ is torsion free HT. Thus, the class of factors with torsion free HT core is well behaved to amplifications and tensor products, producing more examples from the ones described in 1 and 2 above.

The main result in this paper shows that II_1 factors in the class $\mathcal{HT}_{tf}$ have unique decomposition (up to unitary conjugacy) around their torsion free HT cores:

Theorem. *If $N_1, N_2 \subset M$ are torsion free* HT *inclusions of* II_1 *factors then there exists a unitary element $u \in M$ such that $uN_1u^* = N_2$. Thus, factors in the class $\mathcal{HT}_{tf}$ have unique (up to unitary conjugacy) torsion free* HT *core.*

The case of interest is when $M \in \mathcal{HT}_{tf}$ are cross-product factors over their torsion free HT cores, when the above statement becomes:

Corollary 1. *Let N_i be* II_1 *factors, Γ_i be torsion free groups with Haagerup property and σ_i be properly outer cocycle actions of Γ_i on N_i such that $N_i \subset N_i \rtimes_{\sigma_i} \Gamma_i$ are rigid inclusions, $i = 1, 2$. If $N_1 \rtimes_{\sigma_1} \Gamma_1 \simeq N_2 \rtimes_{\sigma_2} \Gamma_2$, via some isomorphism θ, then there exists a unitary element $u \in N_2 \rtimes_{\sigma_2} \Gamma_2$ such that $\mathrm{Ad}(u) \circ \theta$ takes N_1 onto N_2, implements an isomorphism between Γ_1, Γ_2 and cocycle conjugates σ_1, σ_2.*

In particular, Corollary 1 shows that non-isomorphic torsion-free groups Γ give rise to non-isomorphic factors $N \rtimes \Gamma$ in the class $\mathcal{HT}_{tf}$. For the irrational rotation HT factors in Example 1, this gives: If Γ_1, Γ_2 are torsion free non-amenable subgroups of $SL(2, \mathbb{Z})$ and $\Gamma_1 \not\simeq \Gamma_2$ then $M_{\alpha_1}(\Gamma_1) \not\simeq M_{\alpha_2}(\Gamma_2)$, $\forall \alpha_1, \alpha_2$. The Corollary also shows that if N_i are property (T) II_1 factors with aperiodic automorphisms $\beta_i, i = 1, 2$, then $N_1 \rtimes_{\beta_1} \mathbb{Z} \simeq N_2 \rtimes_{\beta_2} \mathbb{Z}$ iff $N_1 \simeq N_2$ and β_1 cocycle conjugate to β_2 (cf. Example 2).

By the Theorem above, classical invariants for the factors $M \in \mathcal{HT}_{tf}$, such as the automorphism group $\mathrm{Out}(M)$ or the fundamental group $\mathscr{F}(M)$, coincide with their

"relative" versions, $\mathrm{Out}(N \subset M) = \mathrm{Aut}(N \subset M)/\{\mathrm{Ad}(u) \mid u \in \mathcal{N}_M(N)\}$ and respectively $\mathscr{F}(N \subset M) = \{t > 0 \mid (N \subset M)^t \simeq (N \subset M)\}$, whenever $N \subset M$ is a torsion free HT core for M.

Moreover, if we denote by $\mathcal{G}_N$ the group of automorphisms of M generated by $\mathrm{Int}(M)$ and by the automorphisms that leave N pointwise fixed ($\simeq \hat{\Gamma}$ when $M = N \rtimes \Gamma$, by [JP]), then by (4.4 of [P1]) $\mathcal{G}_N$ is open and closed in $\mathrm{Aut}(M)$ so the quotient group $\mathrm{Aut}(M)/\mathcal{G}_N$ is countable. By the Theorem, this quotient group is an isomorphism invariant for the factors in the class $\mathcal{HT}_{tf}$. We denote this invariant by $\mathrm{Out}_{HT}(M)$, for $M \in \mathcal{HT}_{tf}$ (same notation as for its analogue invariant for the class $\mathcal{HT}$ in [P1]).

Since $N \subset M$ torsion free HT implies $N \overline{\otimes} N \subset M \overline{\otimes} M$ torsion free HT, and since a choice of $\theta^t : M \simeq M^t$ for each $t \in \mathscr{F}(M)$ gives a one to one embedding $\{\theta^t \otimes \theta^{1/t} \mid t \in \mathscr{F}(M)\} \subset \mathrm{Out}_{HT}(M \overline{\otimes} M)$ (cf. [C1]), it follows that $\mathscr{F}(M)$ is countable as well.

The Theorem shows that if $N \subset M = N \rtimes_\sigma \Gamma$ is a torsion free HT inclusion then any automorphism θ of M (respectively of $M^\infty \overset{\text{def}}{=} M \overline{\otimes} \mathcal{B}(\ell^2\mathbb{N})$), can be perturbed by an automorphism in $\mathcal{G}_N$ (resp. in the group of automorphisms of M^∞ that are amplifications of automorphisms in $\mathcal{G}_N$) to an automorphism θ' that takes the core N (resp. N^∞) onto itself. Thus, $\theta'_{|N}$ (resp. $\theta'_{|N^\infty}$) must lie in the normalizer $\mathcal{N}(\sigma)$ of $\sigma(\Gamma)$ in $\mathrm{Out}(N)$ (resp. in the normalizer $\mathcal{N}^\infty(\sigma)$ of $\sigma(\Gamma)$ in $\mathrm{Out}(N^\infty)$). Since conversely any automorphism in $\mathcal{N}$ implements an automorphism of $N \rtimes_\sigma \Gamma$, it follows that $\mathrm{Out}_{HT}(M)$ is isomorphic to $\mathcal{N}(\sigma)/\sigma(\Gamma)$ and $\mathcal{F}(M)$ is isomorphic to the image via Mod of $\mathcal{N}^\infty(\sigma)$, i.e., $\mathcal{F}(M) = \mathrm{Mod}(\mathcal{N}^\infty(\sigma))$.

Moreover, Corollary 1 implies that $\mathcal{C}(\sigma) \overset{\text{def}}{=} \sigma(\Gamma)' \cap \mathrm{Out}(N)$ and the group $\mathcal{O}(\sigma)$ of outer automorphisms of the group Γ implemented by elements in $\mathcal{N}(\sigma)$ are isomorphism invariants for $N \rtimes_\sigma \Gamma \in \mathcal{HT}_{tf}$. Altogether, we have:

Corollary 2. *If M is a II_1 factor with torsion free* HT *core $N \subset M$ then $\mathscr{F}(M) = \mathscr{F}(N \subset M)$, $\mathrm{Out}(M) = \mathrm{Out}(N \subset M)$. Moreover, $\mathscr{F}(M)$ and $\mathrm{Out}_{HT}(M)$ are countable. If in addition $M = N \rtimes_\sigma \Gamma$, then $\mathrm{Out}_{HT} = \mathcal{N}(\sigma)/\sigma(\Gamma)$, $\mathscr{F}(M) = \mathrm{Mod}(\mathcal{N}^\infty(\sigma))$. Also, $\mathcal{C}(\sigma), \mathcal{O}(\sigma)$ are isomorphism invariants for M and are countable.*

To prove the Theorem, we need two lemmas. The first one shows that in order for the irreducible, torsion free subfactors $N_1, N_2 \subset M$ to be conjugate in M, it is sufficient to have a finite dimensional "intertwining" bimodule between them.

Lemma 1. *Let $N_1, N_2 \subset M$ be irreducible subfactors (i.e., $N_i' \cap M = \mathbb{C}, i = 1, 2$). Assume $N_2 \subset M$ is torsion free. If there exists a non-zero, finite dimensional $N_1 - N_2$ sub-bimodule of $L^2(M)$, then there exists $u \in \mathcal{U}(M)$ such that $uN_1u^* \subset N_2$ with $[N_2 : uN_1u^*] < \infty$. If in addition $N_1 \subset M$ is torsion free as well, then any u as above must satisfy $uN_1u^* = N_2$.*

Proof. This can be easily derived from (2.1 in [P2]), but we'll give here a self contained argument. Let $\mathcal{H} \subset L^2(M)$ be so that $\dim_{N_1}\mathcal{H}$, $\dim\mathcal{H}_{N_2} < \infty$. By taking a submodule

of $\mathcal{H}$ if necessary, we may assume $\mathcal{H}$ is irreducible. Thus, $N_1 \subset JN_2J' \cap \mathcal{B}(\mathcal{H})$ is an irreducible inclusion of finite index. Equivalently, if we denote by p the projection of $L^2(M)$ onto $\mathcal{H}$ then $p \in N_1' \cap \langle M, N_2 \rangle$, $Tr(p) < \infty$ and the inclusion $N_1 p \subset p\langle M, N_2\rangle p$ is irreducible with finite index.

Since N_1 is of type II_1, there exists $q_1 \in \mathcal{P}(N_1), q_1 \neq 0$, such that $Tr(pq_1) \leq 1$. Since $Tr(e_{N_2}) = 1$, it follows that pq_1 is majorized by e_{N_2} in the type II factor $\langle M, N_2\rangle$. But $e_{N_2}\langle M, N_2\rangle e_{N_2} = N_2 e_{N_2}$, so there exists $q_2 \in N_2$ and a partial isometry $V \in \langle M, N_2\rangle$ such that $V^*V = pq_1$, $VV^* = e_{N_2}q_2$. By spatiality, $V(q_1N_1q_1p)V^*$ is an irreducible subfactor of finite index in $q_2N_2q_2e_{N_2}$.

Let $Q \subset N_2$ be such that $q_2 \in Q$ and $q_2Qq_2e_{N_2} = V(q_1N_1q_1p)V^*$ and denote by $\theta : q_1N_1q_1 \simeq q_2Qq_2$ the isomorphism satisfying $VxV^* = \theta(x)e_{N_2}$. Equivalently, $Vx = \theta(x)V, \forall x \in q_1N_1q_1$. By applying the canonical operator valued weight Φ of $\langle M, N_2\rangle$ onto M, it follows that $\xi = \Phi(V)$, which apriorically lies in $L^2(M)$, satisfies $e_{N_2}\xi = V$ and $\xi x = \theta(x)\xi, \forall x \in q_1N_1q_1$. Thus $\xi^*\xi x = \xi^*\theta(x)\xi = x\xi^*\xi, \forall x$, and since $(q_1N_1q_1)' \cap q_1Mq_1 = \mathbb{C}q_1$, it follows that ξ is a scalar multiple of a partial isometry $v \in M$ with $v^*v = q_1$.

Similarly, the intertwiner relation also gives $vv^* \in (q_2Qq_2)' \cap q_2Mq_2$. But by the torsion freeness of $N_2 \subset M$ we have $(q_2Qq_2)' \cap q_2Mq_2 = \mathbb{C}q_2$. Thus, $v(q_1N_1q_1)v^* = q_2Qq_2 \subset q_2N_2q_2$ and since both N_1, N_2 are factors, there exists a unitary element $u \in M$ with $uq_1 = v$ and $uN_1u^* \subset N_2$.

Finally, if $N_1 \subset M$ is torsion free as well, then let $Q_1 \subset N_1$ be a "downward basic construction" for $N_1 \subset u^*N_2u$ (cf. [J]). It follows that $Q_1' \cap N_1 = \mathbb{C}$ but $Q_1' \cap M \supset Q_1' \cap u^*N_2u \neq \mathbb{C}$ unless $N_1 = u^*N_2u$. $\square$

Lemma 2. *Let M be a separable II_1 factor. Assume $B_1, B_2 \subset M$ are von Neumann subalgebras such that M has the property* H *relative to B_1 and $B_2 \subset M$ is rigid. Then B_2 is discrete over B_1, i.e., $L^2(M)$ is generated by $B_2 - B_1$ bimodules which are finite dimensional over B_1.*

Proof. By the property H of M relative to B_1 there exist regular, completely positive, B_1-bimodular maps ϕ_n on M such that $\phi_n \to id_M$ and $T_{\phi_n} \in \mathcal{J}_0(\langle M, B_1\rangle)$. By the rigidity of $B_2 \subset M$ it follows that $\varepsilon_n = \sup\{\|\phi_n(u) - u\|_2 \mid u \in \mathcal{U}(B_2)\} \to 0$. Fix $x \in M$ and note that by (Corollary 1.1.2 of [P1]) we have

$$\|u^*T_{\phi_n}u(\hat{x}) - \hat{x}\|_2 = \|\phi_n(ux) - ux\|_2$$

$$\leq \|\phi_n(ux) - u\phi_n(x)\|_2 + \|\phi_n(x) - x\|_2 \leq 2\varepsilon_n^{1/2} + \|\phi_n(x) - x\|_2.$$

Thus, by taking weak limits of appropriate convex combinations of elements of the form $u^*T_{\phi_n}u$ with $u \in \mathcal{U}(B_2)$, and using (Proposition 1.3.2 of [P1]), it follows that $T_n = \mathcal{E}_{B_2' \cap \langle M, B_1\rangle}(T_{\phi_n}) \in K_{T_{\phi_n}} \cap (B_2' \cap J_0(\langle M, B_1\rangle))$ satisfy $\lim_{n\to\infty} \|T_n(\hat{x}) - \hat{x}\|_2 = 0$. But $x \in M$ was arbitrary. This shows that the right supports of T_n span all the identity

of $\langle M, B_1\rangle$. Since T_n are compact, this shows that $B_2' \cap \langle M, B_1\rangle$ is generated by finite projections of $\langle M, B_1\rangle$. Equivalently, B_2 is discrete relative to B_1. $\square$

Proof of the Theorem. By Lemma 2, N_1 is discrete over N_2 and N_2 is discrete over N_1. Thus, $L^2(M)$ is generated by finite dimensional $N_1 - N_2$ bimodules. By the torsion freeness of $N_1, N_2 \subset M$ and Lemma 1, this implies N_1, N_2 are unitary conjugate. $\square$

We'll now discuss in more details the torsion freeness condition. In particular, we prove that in the case of cross product inclusions, this condition amounts to the group involved being torsion free.

First recall some terminology from ([P3,1]): The *quasi-normalizer* of a subfactor $N \subset M$ is the set $q\mathcal{N}_M(N) = \{x \in M \mid \dim_N L^2(NxN)_N < \infty\}$. Note that the linear span of $q\mathcal{N}_M(N)$ is a $*$-subalgebra of M containing N. Also, note that if $Q \subset N$ is a subfactor with $[N : Q] < \infty$ then $q\mathcal{N}_M(Q) = q\mathcal{N}_M(N)$ (see [P1]). N is *quasi-regular* (or *discrete*) in M if $q\mathcal{N}_M(N)'' = M$.

A typical example of quasi-regular subalgebras is when $N' \cap M = \mathbb{C}$ and N is *regular* in M, i.e., $\mathcal{N}_M(N)'' = M$, equivalently when $M = N \rtimes_\sigma \Gamma$ for some properly outer cocycle action of the group $\Gamma = \mathcal{N}(N)/\mathcal{U}(N)$ on N. Other examples are the symmetric enveloping inclusions associated to extremal subfactors of finite Jones index ([P3]).

Note that by (3.4 in ([P1]), if M has the property H relative to a subfactor $N \subset M$ then N is quasi-regular in M. Thus, a HT inclusion of factors $N \subset M$ as in the Theorem is automatically quasi-regular.

Proposition. 1°. *Let $N \subset M$ be an irreducible inclusion of factors. If $q\mathcal{N}_M(N)'' = \mathcal{N}_M(N)''$ and we denote $\Gamma = \mathcal{N}_M(N)/\mathcal{U}(N)$, then $N \subset M$ is torsion free iff Γ is torsion free.*

2°. *If N is quasi-regular in M and $N \subset M$ is torsion free then there exist no intermediate subfactors $N \subset P \subset M$ such that $[P : N] < \infty$, $P \neq N$.*

Proof. Indeed, for if $u_0 \in \mathcal{N}_M(N)$ implements an automorphism θ_0 with outer period $2 \leq k < \infty$ then by Connes' theorem there exists $v \in \mathcal{U}(N)$ such that $(vu_0)^k = 1$. Thus, if $\theta = \mathrm{Ad}(vu_0)$ then $Q = N^\theta$ satisfies $Q' \cap N = \mathbb{C}$ while $vu_0 \in (Q' \cap M) \setminus \mathbb{C}1$.

Conversely, let Q be an irreducible subfactor of N with finite index. Assume $a \in (Q' \cap M) \setminus \mathbb{C}1$. Then $a \in q\mathcal{N}_M(Q) = q\mathcal{N}_M(N)$ (the equality holds because Q has finite index in N). By hypothesis, it follows that $a = \Sigma_g x_g u_g$ for some $x_g \in N$, where $u_g \in \mathcal{N}_M(N)$ are some unitaries implementing the cross-product construction $N \subset N \rtimes \Gamma = \mathcal{N}_M(N)''$, where $\Gamma = \mathcal{N}_M(N)/\mathcal{U}(N)$. This implies that there exists $g \neq e$ such that $x_g \neq 0$ and $yx_g = x_g\sigma_g(y), \forall y \in Q$. By $Q' \cap N = \mathbb{C}$ it follows that the partial isometry in the polar decomposition of x_g is a unitary element v with $\theta = \mathrm{Ad}(vu_g)$ satisfying $Q \subset N^\theta$. Since $[N : Q] < \infty$, this shows that θ is periodic, thus g has torsion.

2°. If there exists $N \subset P \subset M$ with $[P : N] < \infty$ and $P \neq N$ then let $Q \subset N$ be a

downward basic construction for $N \subset P$ (cf. [J]). We then have $Q' \cap N = \mathbb{C}$ (because $N' \cap P = N' \cap M = \mathbb{C}$) but $Q' \cap P \neq \mathbb{C}$, as it contains the Jones projection. $\square$

Remarks. 1°. The converse implication in part 2° of the Proposition is probably true as well. This is of course the case when $M = N \rtimes_\sigma \Gamma$, by part 1° of that statement.

2°. By Corollary 1, isomorphism of cross product HT factors $M = N \rtimes_\sigma \Gamma$ (such as the irrational rotation HT factors $M_\alpha(\Gamma)$), with torsion free Γ, amounts to the isomorphism of the groups Γ and the cocycle conjugacy of the (cocycle) actions σ. But in both Examples 1 and 2, we could not find ways to completely distinguish between the (cocycle) actions σ. In particular, we could not obtain precise calculations of $\text{Out}(M)$, $\mathscr{F}(M)$ (or for that matter $\mathcal{C}(\sigma)$, $\mathcal{O}(\sigma)$) by the method of calculation of Corollary 2. However, a classification (= non-isomorphism) "modulo countable sets" of the factors $M_\alpha(\Gamma)$ is obtained in ([NPS]). Note that these factors (and in fact all non-McDuff factors of the form $N \rtimes_\sigma \Gamma$ with $N \simeq R$ and Γ in the class $\mathcal{C}$ of [O]) follow prime by Ozawa's recent results in ([O]).

3°. Related to Example 2, one can prove the following statement, by using an argument similar to the proof of (Proposition 9 in [GP]): If (N, τ) is a finite von Neumann algebra with a faithful normal trace state and σ is a properly outer cocycle action of an amenable group Γ on (N, τ), then the inclusion $(N \subset N \rtimes_\sigma \Gamma)$ is rigid if and only if N has the property (T) in ([P1]), i.e., iff $N \subset N$ is rigid. This notion of property (T) coincides with the one considered in ([Jol]) in the case the algebras are of the form $N = L_\mu(H)$, for H a discrete group and μ a cocycle on it, when in fact both conditions are equivalent to the property (T) for H. They also coincide in the case N has finite dimensional center, when they are equivalent to the original Connes-Jones definition in ([CJ]).

4°. It is interesting to know whether the free group factors $L(\mathbb{F}_n)$ can be realized as "cores" of HT (or merely rigid) inclusions $L(\mathbb{F}_n) \subset M$. It may be that this happens iff n is finite. The affirmative answer to the "if" part of this problem alone (i.e., showing that $L(\mathbb{F}_2)$ can be realized as a rigid core), could provide new insight to the "(non)isomorphism of the free group factors" problem. In this respect, we should mention that it is an open problem whether $\mathbb{F}_n \subset \text{Aut}(\mathbb{F}_n)$ has the relative property (T) ([M]) for some $2 \leq n < \infty$. In particular, it is not known whether $\mathbb{F}_2 \subset \mathbb{F}_2 \rtimes SL(2, \mathbb{Z})$ has the relative property (T) or not.

References

[Bu] M. Burger, *Kazhdan constants for $SL(3, \mathbb{Z})$*, J. reine angew. Math., **413** (1991), 36-67.

[C1] A. Connes: *A type II_1 factor with countable fundamental group*, J. Operator Theory **4** (1980), 151-153.

[C2] A. Connes: *Classification des facteurs*, Proc. Symp. Pure Math. **38** (Amer. Math.

Soc., 1982), 43-109.

[CJ] A. Connes, V.F.R. Jones: *Property* (T) *for von Neumann algebras*, Bull. London Math. Soc. **17** (1985), 57-62.

[GP] D. Gaboriau, S. Popa: *An Uncountable Family of Non Orbit Equivalent Actions of* $\mathbb{F}_n$, preprint, math.GR/0306011.

[dHV] P. de la Harpe, A. Valette: "La propriété T de Kazhdan pour les groupes localement compacts", Astérisque **175**, Soc. Math. de France (1989).

[Jol] P. Jolissaint: *Property* (T) *for discrete groups in terms of their regular representation.*

[J] V.F.R. Jones : *Index for subfactors*, Invent. Math. **72** (1983), 1-25.

[JP] V.F.R. Jones, S. Popa: *Some properties of MASAs in factors*, in "Invariant subspaces and other topics", pp. 89-102, Operator Theory: Adv. Appl. **6**, Birkhuser, 1982.

[K] D. Kazhdan: *Connection of the dual space of a group with the structure of its closed subgroups*, Funct. Anal. and its Appl. **1** (1967), 63-65.

[M] G. Margulis: *Finitely-additive invariant measures on Euclidian spaces*, Ergodic. Th. and Dynam. Sys. **2** (1982), 383-396.

[MvN] F. Murray, J. von Neumann: *Rings of operators IV*, Ann. Math. **44** (1943), 716-808.

[NPS] R. Nicoara, S. Popa, R. Sasyk : *Some remarks on irrational rotation HT factors*, preprint, math.OA/0401...

[O] N. Ozawa: *A Kurosh type theorem for* II_1 *factors*, preprint, math.OA/0401121.

[P1] S. Popa: *On a class of type* II_1 *factors with Betti numbers invariants*, preprint OA/0209130.

[P2] S. Popa: *Strong Rigidity of* II_1 *Factors Coming from Malleable Actions of Weakly Rigid Groups, part I*, preprint, math.OA/0305306.

[P3] S. Popa: *Some properties of the symmetric enveloping algebras with applications to amenability and property* T, Documenta Math. **4** (1999), 665-744.

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